$+ \begin{pmatrix} 10 \\ 7 \end{pmatrix} \times x^7 + \begin{pmatrix} 10 \\ 6 \end{pmatrix} \times x^6 + \begin{pmatrix} 10 \\ 5 \end{pmatrix} \times x^5 + \begin{pmatrix} 10 \\ 4 \end{pmatrix} \times x^4 + \begin{pmatrix} 10 \\ 3 \end{pmatrix} \times x^3 + \begin{pmatrix} 10 \\ 2 \end{pmatrix} \times x^2 + \begin{pmatrix} 10 \\ 1 \end{pmatrix} \times x^1 + \begin{pmatrix} 10 \\ 0 \end{pmatrix} \times x^0.$

This is, of course, a bit cumbersome to write, so writing this polynomial in sigma form makes sense.

Therefore, $(x + 1)^{10} = \sum_{k=0}^{10} \begin{pmatrix} 10 \\ k \end{pmatrix} \times x^k$.

This is a very nice generalization. Let's try another expansion like this, but one that is a little more complicated, to see if the same type of mathematical structure applies.

EXAMPLE 2.3c: What is $(2x + 5)^{15}$?

SOLUTION: Rather than write out all fifteen terms of $2x + 5$, we'll imagine them written out in a long line. As in Example 2.3b, each term in the product is generated by selecting either $2x$ or 5 from each of the fifteen terms and multiplying them together.

So, $(2x) \times 5 \times (2x) \times (2x) \times (2x) \times 5 \times (2x) \times (2x) \times 5 \times (2x) \times (2x) \times (2x) \times 5 \times (2x) \times (2x)$ is one possible term in the expansion, and this term would be grouped with the other x^{11} terms. Let's go through a few powers of x and try to determine the coefficients.

The highest power of x will be x^{15}, and this will occur when the $2x$ is selected from all fifteen terms. $(2x)^{15} = 2^{15} \times x^{15}$, and so the coefficient of x^{15} will be 2^{15}, or 32,768.

The next highest power of x is x^{14}, and these will be generated by selecting the $2x$ from every term except one, from which a 5 will be selected. Each of these terms will therefore look like:

$(2x) \times (2x) \times (2x) \times (2x) \times (2x) \times (2x) \times (2x) \times (2x) \times (2x) \times (2x) \times (2x) \times (2x) \times (2x) \times (2x) \times 5$,

and so each will equal $(2x)^{14} \times 5 = 2^{14} \times 5 \times x^{14}$, or $81920 \times x^{14}$. But how many of these terms will be created? With fifteen spots and needing to select 14 of them as $2x$ (or one of them as 5), there should be 15 ways to do this. Note that $\begin{pmatrix} 15 \\ 14 \end{pmatrix} = \begin{pmatrix} 15 \\ 1 \end{pmatrix} = 15$, because selecting 14 spots for the $2x$ automatically selects the spot for the 5, and vice versa. Therefore, the final coefficient of x^{14} will be $81{,}920 \times 15 = 1{,}228{,}800$.

Sometimes the numerical values disguise rather than assist generalization. Instead of thinking about this as $1228800 \times x^{14}$, let's keep track of where it came from: $\begin{pmatrix} 15 \\ 14 \end{pmatrix} \times (2x)^{14} \times 5$. Again, this calculation makes sense: when selecting $2x$ from 14 of the 15 terms, $2x$ will be raised to the 14th power, and 5 will be selected from the last term. This selection can then occur in $\begin{pmatrix} 15 \\ 14 \end{pmatrix}$ ways, since there are 15 terms, and we are selecting the $2x$ from 14 of them.

What about x^{13} ? These are generated by selecting the $2x$ from 13 of the 15 terms. For example:

$(2x) \times 5 \times (2x) \times (2x) \times (2x) \times (2x) \times (2x) \times 5 \times (2x) \times (2x) \times (2x) \times (2x) \times (2x) \times (2x) \times (2x)$. Each of these terms will therefore equal $(2x)^{13} \times 5^2$. But how many of these terms will there be? With 15 spots, selecting 13 of them to be $2x$ needs to occur without replacement, and the order of selection does not matter. So this can